Lecture 2. Linear systems and matrices

Def (1) A matrix is a rectangular array of numbers.

(2) For a matrix with m rows and n columns, its size is $m \times n$.

e.g. $\begin{bmatrix} 2 & 3 & 2 \\ -1 & 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 3 & -1 & 4 & 5 \\ 2 & 0 & 1 & 0 \\ 0 & 3 & -2 & 6 \end{bmatrix}$

2×3 matrix $\quad$ 3×4 matrix

Note A linear system can be represented by a matrix with coefficients and constant terms as entries.

e.g. $\begin{cases} x_1 - 2x_2 + 3x_3 = 2 \\ \quad\quad 3x_2 - 7x_3 = 4 \\ 5x_1 \quad\quad - 2x_3 = 6 \end{cases} \rightsquigarrow \begin{cases} 1 \cdot x_1 - 2x_2 + 3x_3 = 2 \\ 0 \cdot x_1 + 3x_2 - 7x_3 = 4 \\ 5x_1 + 0 \cdot x_2 - 2x_3 = 6 \end{cases}$

$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 3 & -7 & 4 \\ 5 & 0 & -2 & 6 \end{array}\right]$ "augmented matrix"

coefficients $\quad$ constants

* The size of the matrix is given as follows:

- number of rows = number of equations
- number of columns = number of variables + 1

column for constant terms

Def A matrix is in reduced row echelon form (RREF) if it has the following properties:

(i) All nonzero rows are above any zero rows.

(ii) The leading nonzero entry in each row is 1.

(iii) Each leading 1 is the only nonzero entry in its column.

(iv) Each leading 1 is to the right of the leading 1 in the row above it.

e.g. $\begin{bmatrix} 1 & 0 & 3 & 0 & -2 \\ 0 & 1 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ * circled entries are leading 1s

staircase pattern

Thm Every matrix A can be simplified to a unique reduced row echelon form, denoted by RREF(A), through a sequence of the following operations:

- adding to one row a multiple of another row
- interchanging two rows
- multiplying one row by a nonzero constant

Note (1) These operations are called elementary row operations.

(2) We can solve a linear system by simplifying its matrix to an RREF.

Ex Find the solution of the linear system

$$\begin{cases} x_1 \qquad\quad - 3x_3 = 8 & (\text{Eq. } 1) \\ 2x_1 + 3x_2 + 9x_3 = 10 & (\text{Eq. } 2) \\ \qquad\quad x_2 + 2x_3 = 1 & (\text{Eq. } 3) \end{cases}$$

Sol 1 (Algebra)

We first aim to get a system which does not involve x_1.

(Eq. 2) − 2 (Eq. 1) : $(2x_1 + 3x_2 + 9x_3) - 2(x_1 - 3x_3) = 10 - 2 \cdot 8$

$$\Rightarrow \cancel{2x_1} + 3x_2 + 9x_3 - \cancel{2x_1} + 6x_3 = -6 \quad (x_1 \text{ eliminated})$$

$$\Rightarrow 3x_2 + 15x_3 = -6 \quad (\text{Eq. } 2R)$$

(Eq. 2R) and (Eq. 3) depend on 2 variables : x_2 and x_3

We can solve them using the idea from Lecture 1.

$\frac{1}{3}$(Eq. 2R) : $\frac{1}{3}(3x_2 + 15x_3) = \frac{1}{3} \cdot (-6) \Rightarrow x_2 + 5x_3 = -2$ (Eq. 2RR)

coefficients 1

(Eq. 3) − (Eq. 2RR) : $(x_2 + 2x_3) - (x_2 + 5x_3) = 1 - (-2)$

$$\Rightarrow \cancel{x_2} + 2x_3 - \cancel{x_2} - 5x_3 = 3 \quad (x_2 \text{ eliminated})$$

$$\Rightarrow -3x_3 = 3 \Rightarrow \underline{x_3 = -1}$$

(Eq. 2RR) : $x_2 + 5x_3 = -2 \Rightarrow x_2 + 5 \cdot (-1) = -2 \Rightarrow x_2 = 3$

(Eq. 1) : $x_1 - 3x_3 = 8 \Rightarrow x_1 - 3 \cdot (-1) = 8 \Rightarrow x_1 = 5$

Hence the solution is given by $\boxed{x_1 = 5,\ x_2 = 3,\ x_3 = -1}$

Sol 2 (Matrices)

We represent the system by a matrix and simplify it to an RREF.

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 2 & 3 & 9 & 10 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

In column 1, all entries other than the leading 1 must be 0

Row 2 − 2·Row 1: $[2\ 3\ 9\ 10] - 2[1\ 0\ -3\ 8] = [0\ 3\ 15\ -6]$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 3 & 15 & -6 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

In row 2, the leading nonzero entry must be 1

$\frac{1}{3}\cdot$Row 2: $\frac{1}{3}[0\ 3\ 15\ -6] = [0\ 1\ 5\ -2]$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

In column 2, all entries other than the leading 1 must be 0

Row 3 − Row 2: $[0\ 1\ 2\ 1] - [0\ 1\ 5\ -2] = [0\ 0\ -3\ 3]$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & -3 & 3 \end{array}\right]$$

In row 3, the leading nonzero entry must be 1

$-\frac{1}{3}\cdot$Row 3: $-\frac{1}{3}[0\ 0\ -3\ 3] = [0\ 0\ 1\ -1]$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

In column 3, all entries other than the leading 1 must be 0

Row 1 + 3·Row 3: $[1\ 0\ -3\ 8] + 3[0\ 0\ 1\ -1] = [1\ 0\ 0\ 5]$

Row 2 − 5·Row 3: $[0\ 1\ 5\ -2] - 5[0\ 0\ 1\ -1] = [0\ 1\ 0\ 3]$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

RREF!

The final matrix yields the solution $\boxed{x_1 = 5,\ x_2 = 3,\ x_3 = -1}$

Ex Find the solution of the linear system

$$\begin{cases} x_1 - 2x_2 + 3x_3 = 4 \\ 2x_1 - 3x_2 + x_3 = 3 \\ 3x_1 - 4x_2 - x_3 = 2 \end{cases}$$

Sol We represent the system by a matrix and simplify it to an RREF.

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 4 \\ 2 & -3 & 1 & 3 \\ 3 & -4 & -1 & 2 \end{array}\right]$$

In column 1, all entries other than the leading 1 must be 0

Row 2 − 2·Row 1 : $[2\ {-3}\ 1\ 3] - 2[1\ {-2}\ 3\ 4] = [0\ 1\ {-5}\ {-5}]$

Row 3 − 3·Row 1 : $[3\ {-4}\ {-1}\ 2] - 3[1\ {-2}\ 3\ 4] = [0\ 2\ {-10}\ {-10}]$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & 4 \\ 0 & 1 & -5 & -5 \\ 0 & 2 & -10 & -10 \end{array}\right]$$

In column 2, all entries other than the leading 1 must be 0

Row 1 + 2·Row 2 : $[1\ {-2}\ 3\ 4] + 2[0\ 1\ {-5}\ {-5}] = [1\ 0\ {-7}\ {-6}]$

Row 3 − 2·Row 2 : $[0\ 2\ {-10}\ {-10}] - 2[0\ 1\ {-5}\ {-5}] = [0\ 0\ 0\ 0]$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -7 & -6 \\ 0 & 1 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

RREF!

The final matrix yields the equations

$$\begin{cases} x_1 - 7x_3 = -6 \\ x_2 - 5x_3 = -5 \end{cases} \Rightarrow \begin{cases} x_1 = 7x_3 - 6 \\ x_2 = 5x_3 - 5 \end{cases}$$

x_3 can take any value t

$\Rightarrow$ $\boxed{x_1 = 7t - 6,\ x_2 = 5t - 5,\ x_3 = t \text{ with } t \text{ arbitrary}}$

Note x_3 is called a free variable for being free to take any value.

A free variable corresponds to a column without a leading 1.